PROCESS OPTIMIZATION

Introduction: A task that a process control engineer may face is to work out the optimum steady-state operating point of an item of equipment, or even of the whole process. Technical questions in optimization focus on:

- how to select the best overall operating point;
- how to select the best achievable operating point within some constraints;
- how to formulate and solve the optimization problem.

Process optimization is an important aspect of process operation and control because it has a considerable impact on profit. This article illustrates the practice and theory of process optimization using optimal operation of a compressor as an example. The reason for choosing a compressor is that the characteristic curves on the compressor map are easy to visualize, and they help to make the ideas of optimization become intuitive.

Compressor optimization: A compressor map shows pressure ratio on the vertical axis and flow rate on the horizontal axis. Contours such as those in Figure 1 show the compressor efficiency at the different pressure/flow operating points. To operate the compressor of Figure 1 at maximum efficiency, roughly in the middle of the smallest oval, would require the pressure ratio to be 1.8 and the volumetric flow rate to be just over 50% of the maximum flow rate to achieve this optimum performance.

A second map (Figure 2) shows the combinations of pressure and volumetric flow rate which can be achieved in the compressor when it operates at different rotation speeds. Each line in this map represents operation at a constant rotation speed and it shows what combinations of pressure ratio and flow rate will be possible at that rotation rate. If the compressor is to be operated at the point marked \( x \) where its efficiency is greatest, then the rotation speed must be about 75% of full speed because \( x \) is half way between the 70% and 80% speed lines. Once the speed, pressure ratio and volumetric flow rate have been selected then the optimization is done. The efficiency of the compressor at this operating point is more than 78%.

A practical compressor map shows the efficiency contours and speed characteristics together on the same figure, as in Figure 3 which also shows the surge line. The part of the map to the left of the surge line is an unstable region where the compressor should not be operated.
Mathematical formulation: Mathematical optimization has a formal notation. The problem is to find the solutions (e.g. the operating parameters: speed, flow rate and pressure ratio) which maximise an objective function (e.g. maximise the efficiency). The solutions may have to obey some constraints, for instance they have to fit the speed, pressure ratio and flow rate relationships in Figure 2.

The efficiency contours in the map have the mathematical form of an objective function \( \eta = f(P_R, v, n) \), which is a compact way of expressing the fact that \( \eta \) (efficiency) depends on \( P_R \) (pressure ratio), \( v \) (flow rate) and \( n \) (rotation speed). As the compressor example shows, it is not necessary to have an analytical expression for the function. A graphical representation or a look-up table can provide the necessary information about the function if no mathematical model is available. Since the efficiency is to be maximised, the mathematical problem to be solved is:

\[
\max_{P_R, v, n} \eta = f(P_R, v, n)
\]

i.e. find the maximum efficiency and the values of \( P_R, v \) and \( n \) which give maximum efficiency.

The statement is not complete, however, because the optimization must also pay attention to the fact that the speed characteristics only allow certain combinations of pressure ratio, flow rate and rotation speed. The mathematical formulation is an algebraic relationship \( g(P_R, v, n) = 0 \) where \( g \) indicates a function that relates the variables \( P_R, v \) and \( n \) to one another. This is one equation with three unknowns, so any one of \( P_R \), \( v \) or \( n \) can be calculated (or looked up on the map) if the other two are specified. A more complete mathematical expression of the compressor optimization problem is therefore:

\[
\begin{align*}
\max_{P_R, v, n} & \quad \eta = f(P_R, v, n) \\
\text{subject to} & \quad g(P_R, v, n) = 0
\end{align*}
\]

The solution to the above problem is the maximum value of the efficiency, \( \eta \), for this compressor and the values of \( P_R, v \) and \( n \) which provide the maximum efficiency.

Something is still missing, however, which is that the compressor must operate in the stable region to the right of the surge line. The surge line in a compressor is always a hard constraint because operating to the left of the surge line can damage the compressor. In the optimization problem, a stable optimum is achieved by requiring \( P_R, v \) and \( n \) to be in the allowed region, designated by \( A \).

\[
\begin{align*}
\max_{\{P_R, v, n\} \in A} & \quad \eta = f(P_R, v, n) \\
\text{subject to} & \quad g(P_R, v, n) = 0
\end{align*}
\]

The notation \( \{P_R, v, n\} \in A \) indicates that the solutions must lie in region \( A \) to the right of the surge line. Any automated algorithm for finding the optimum must check that solutions meet this condition.

Constrained optimization: It is possible to include additional constraints into an optimization problem. For instance suppose that, even when running flat out, a motor driving the compressor can only achieve 60% of the compressor’s maximum rated speed. In that case the global optimum efficiency at the point marked \( x \) on Figure 4 cannot be achieved because the motor cannot drive the compressor fast enough. The only parts of the map that are accessible with the slow motor are in the shaded region of Figure 4. The constrained optimization problem seeks for the best efficiency that can be achieved within the shaded region. A constrained optimum is often located on the constraint boundary, and the best efficiency that can be achieved in this example is at the point marked \( o \) where the efficiency is 76%. The mathematical formulation for this problem is:

\[
\begin{align*}
\max_{\{P_R, v, n\} \in A} & \quad \eta = f(P_R, v, n) \\
\text{subject to} & \quad g(P_R, v, n) = 0 \\
& \quad n \leq 60\% \text{ of full speed}
\end{align*}
\]

In other constrained cases the flow rate through the compressor might be limited by a bottleneck elsewhere in the process, or there may be a
maximum permitted discharge pressure. These are also examples of constraints that might move the feasible optimum away from the global optimum. They are represented in a similar way, for instance the formulation below states that the flow rate cannot be higher than \( v_b \) where \( v_b \) is the flow rate through a process bottleneck.

\[
\begin{align*}
\text{maximize} & \quad \eta = f(P_v, n, v, i) \\
\text{subject to} & \quad g(P_v, n, v, i) = 0 \\
& \quad v \leq v_b \\
& \quad i = 1, 2
\end{align*}
\]

When the flow rate is constrained, the feasible operating region can become quite small because it is limited to the left by the surge line and to the right by the flow rate constraint, as shown in Figure 5.

![Figure 4. Optimization when the motor speed has an upper bound of 60%](image)

**Compressor restaging.** One way in which the performance of a compressor can be improved is by a procedure called restaging which makes adjustments to the physical configuration of the compressor. One reason why restaging might be undertaken is because the flow rate has decreased permanently. If the flow rate becomes smaller then the efficiency deteriorates and also the compressor operating point gets closer to the surge limit. This is a dangerous place to operate because a small disturbance could cause a surge. The solution is to restage the compressor so that the new surge line is further away from the operating point. Figures 5 and 6 show a compressor map before restaging and after restaging to a lower flow configuration. The compressor map changes as a result of restaging, the operating point becomes more efficient and is further from the new surge line.

**Mixed integer optimization:** Compressor restaging is an example of mixed-integer optimization. As well as the optimization of \( \eta \) by adjustments to the continuous variables \( P_v \) and \( n \), we also have to make a decision about restaging. The mathematical formulation is

\[
\begin{align*}
\text{maximize} & \quad \eta = f(P_v, n, i) \\
\text{subject to} & \quad g(P_v, n, i) = 0 \\
& \quad i = 1, 2
\end{align*}
\]

The above optimization problem calls for \( i \) to be adjusted along with \( P_v \) and \( n \), where \( i \) is an integer decision variable taking values of 1 or 2 representing the two compressor configurations on offer. The second constraint shows that this optimization problem is for a fixed flow rate which requires a solution such that \( v = v_b \). Finally, the notation \( \{P_v, n, i\} \in A(i) \) shows that the allowed solutions depend on the compressor configuration. This happens because the surge line is different in the reconfigured compressor.

**Linear, quadratic and nonlinear programming:** Linear programming (LP) is the name given to an optimization problem in which the objective function and the constraints are linear functions. A QP problem (quadratic programming) describes the case where the constraint equations are linear and the objective function is quadratic. The compressor problem certainly is not linear because the efficiency contours indicate there is a peak in the objective function, so it must at least be a quadratic objective function. The equation \( g(P_v, n, i) = 0 \) is not linear either, because the constant speed lines are curved. Therefore compressor optimization is a nonlinear programming problem (NLP). If restaging is included then it is a mixed integer nonlinear programming problem (MINLP).

**Mathematical algorithms and tools:** There are many algorithms to tackle optimization problems. Some problems allow for exact analytical solutions if the functions \( f \) and \( g \) are simple (e.g. quadratic or linear). Other problems are solved by approximating them with locally linear or quadratic functions. Others have to be tackled by purely numerical means, for instance if the compressor optimization problem were to be automated then the algorithm would require a quick way to search a table holding the compressor map. Wikipedia [1] gives a review of optimization algorithms.

A difficult problem that arises in optimization is when the objective function is not convex. The compressor example is a convex problem because the efficiency contours show there is just one single peak in the objective function.
More complicated objective functions may have multiple peaks and the optimizer then faces the task of finding the global optimum (the highest peak). This can be time consuming and very difficult to do, and often engineers will accept a locally optimum solution which is good enough, even though it may not be the global optimum.

Wikipedia provides a list of commercial and freeware tools and solvers that can be applied to optimization problems [1]. It takes some skill to become proficient in the mathematical formulation of an optimization problem and in deciding the best algorithm for its solution. Optimization based on mathematical models also requires modelling skills.

Reference: